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Comment on "Laminar Stagnation-Point Heat Transfer for a Two-Temperature Argon Plasma"

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IN Ref. 1, the authors attempt to simplify the heat transfer problem by assuming that "the freestream static temperature near the stagnation point is constant along the x direction and is equal to that at the stagnation point." While they considered the Joule heating term to be constant, as shown in Eq. (16), they neglected to balance the global energy in the freestream. If the above assumption is used in the global energy equation, the current density should be zero in the freestream. And if the Joule heating in the boundary layer is constant, as they have assumed, then it should be zero. On the other hand, if the externally applied electric field is a nonzero constant, then there should be additional terms in the energy equations from (18) through (20) pertaining to the freestream temperature variation along the x direction.

It is apparent that a stream function ψ is used so that the continuity equation is satisfied by $\rho u r_w = \partial\psi/\partial y$ and $\rho v r_w = -\partial\psi/\partial x$. However, the stream function is not defined, nor is the velocity component in the y direction defined in terms of the "dimensionless normal velocity." The sign for the term $\rho_0 U_0 \mathrm{d}U_0/\mathrm{d}x$ in Eq. (9) is incorrect.[2] This term is then dropped completely from Eq. (17), which does not, therefore, represent the flow configuration considered in the paper.

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Reply by Authors to H. Chuang

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IN response the Comment, we would like to point out some errors in the paper.[2] First, the freestream velocity, U_0, in Fig. 1 is to be replaced by U_∞, the approaching flow velocity. For an axisymmetric spherical shaped body, the relation between the two in the stagnation region is $U_0 = 3U_\infty x/(2R_w)$. Second, as pointed out in by Chuang, the sign in the first term on the right-hand side of Eq. (9) is positive. Third, in Eq. (4), x_i is replaced by x_e.

The term $\rho_0 U_0 \mathrm{d}U_0/\mathrm{d}x$ in Eq. (9) can easily be transformed to a pressure gradient parameter $\beta = (2s/U_0)(\mathrm{d}U_0/\mathrm{d}x)$, which, as is well-known, has a value of ½ for the present case.[3] This value is used in the third term of Eq. (17). In addition, because of the above freestream velocity distribution in the stagnation region, $U_0 = 0$ as $x \to 0$. Furthermore, at the stagnation point, $\partial h/\partial x = 0$ due to symmetry. Thus, in the freestream the first term in Eqs. (10) and (12) is zero, and it is left to the other y-dependent terms to balance the Joule heating term in the freestream. The obvious conclusion is that the $\partial/\partial y$ terms in these two equations are not zero, unless the Joule heating term in the freestream is balanced by a term due to a dissipative mechanism such as radiation, which has not been included in the present study. However, the problem is not serious, since in a differential equation of second order only two boundary conditions are to be prescribed—the temperatures at $\eta = 0$ and η_{max}—and nonzero values of the temperature gradients in the freestream need not be taken into account. Actual numerical calculation shows that this gradient at the freestream ($\eta = \eta_{max}$) for the current density range studied in the paper[1] is indeed very small.

As Chuang correctly surmises, the transformation of the continuity, momentum, and energy equations from the body-based coordinate system to those in the (s,η) coordinate system requires introduction of stream function and "dimensionless normal velocity." These definitions were thought to be quite standard, and were left out to save space. However, Refs. 2 and 3 will give details about these definitions.

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Comment on "Dynamic Condensation"

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IN Ref. 1, Paz has pointed out some of the shortcomings of static condensation methods such as those of Guyan[2] and Irons[3] when used to reduce the number of degrees of freedom considered in vibration analysis and dynamic response analysis of systems having large numbers of degrees of freedom. He has presented a technique of dynamic condensation that greatly improves the accuracy of both eigenvalues and eigenvectors for such methods. Moreover, in principle, the dynamic condensation method, iteratively applied, can lead to exact solutions for the eigenvalues and eigenvectors of the complete unreduced matrix equation subject only to the usual limitations of computational accuracy.

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However, dynamic condensation greatly increases the amount of computation required, since the method involves repeated inversions of matrices of order equal to the number of degrees of freedom eliminated from the reduced problem. The computational effort required for dynamic condensation at best lies between the effort required for finding all or most of the eigenvalues and eigenvectors of the reduced eigenvalue problem and that required for finding the same number of lowest eigenvalues represented in the reduced problem from the complete matrix equation. The comparison of computational effort depends on the techniques used in extracting the lower eigenvectors and eigenvalues of the complete problem, the number of iterations of dynamic condensation employed, the computer hardware and software available, and the character of the dynamic system being analyzed.

Modifications of static condensation techniques devised by Kidder,[4] Miller,[5] and others provide corrections to the eigenvectors, but not the eigenvalues, obtained from statically condensed eigenvalue analyses. This is evident in Table 1 of Ref. 1, in which the approximate eigenvalues are identical for the static and modified static condensation methods; the errors in the four eigenvalues of the reduced problem are 1.2, 11.3, 30.4, and 36.8%. Since the displacements, shears, and moments in structures subjected to a vibration spectrum, as in Paz's illustrative example, depend on both the eigenvectors and the eigenvalues, the correction of the eigenvalues along with the eigenvectors is appropriate for calculations of these quantities.

The writer[6] has called attention to the fact that the corrections for eigenvectors lead directly to corresponding corrections for eigenvalues, and has suggested several approximate methods for evaluating the eigenvalue correction. A formula for correction of the eigenvalue based on second-order perturbation theory may readily be applied to the same (or higher) order in the expansion of $[K_{22}-\omega^2 M_{22}]^{-1}$ as used for the eigenvectors by Kidder and Miller, where K_{22} and M_{22} are the stiffness and mass matrices of the degrees of freedom eliminated in reduction. Johson et al,[7] who arrived at a form of this correction by other means, have shown in an illustrative example that it can compensate for most of the error in the eigenvalues obtained from the reduced problem. Alternatively, a simple and computationally economic technique for computing eigenvalue corrections is to use the eigenvectors of the reduced problem and the modified eigenvectors for the degrees of freedom eliminated as the eigenvectors of the complete problem and to calculate the eigenvalues from the Rayleigh quotients. Under appropriate conditions on eigenvalue separation (e.g., as discussed by Fox[8]) if the error in the eigenvector is of order ϵ, the error in the eigenvalue will be of order ϵ^2.

Another question raised[6] regarding use of th back-substitution methods of Kidder and Miller to improve eigenvectors concerns the validity of the expansion of $[K_{22}-\omega^2 M_{22}]^{-1}$ in a power series in ω^2. If any eigenvalues of the system represented by $(K_{22}-\omega^2 M_{22})y_2=0$, where y_2 is an eigenvector, lie below a sought-after eigenvalue of the reduced problem, the expansion is invalid and the eigenvectors obtained by modified correction methods are not assuredly an improvement over those resulting from static condensation. Paz's dynamic condensation technique avoids this problem, since he makes no use of matrix series expansions. Nevertheless, low-lying eigenvalues of $(K_{22}-\omega^2 y_{22})y_2=0$ may be present and they may produce eigenvalues in the range of eigenvalues of the reduced problem. These or some eigenvalues of the reduced problem may then be missed using dynamic condensation. Therefore, it is still of interest to establish that all of the eigenvalues of the eliminated system lie above those to be found from the reduced system.

A straightforward method for establishing an approximate lower bound for the eigenvalues of the eliminated system is to form the equation

$$y_2=\omega^2 M_{22}^{-1}K_{22}y_2 \tag{1}$$

The matrix calculations required here are carried out in static condensation (or dynamic condensation with $\omega^2=0$) in any case. Then, since all of the eigenvalues, ω^2, are real and positive, it can easily be seen that

$$1/\omega_1^2 \le \mathrm{tr}(K_{22}^{-1}M_{22}) \tag{2}$$

where ω_1^2 is the lowest eigenvalue satisfying Eq. (1), and tr denotes the trace (sum of the diagonal elements).[9] While this may be too crude a lower bound for ω_1^2 to be satisfactory for some systems, it can show when the eigenvalues of the eliminated system lie well above the range of interest in the reduced system. Condensation methods, including Paz's method, can be used with greater confidence when this is known. An improved estimate can be obtained by calculating a lower bound for ω^4 from the trace of the square of the matrix. Another technique, which can be employed if the lower bound obtained by these is too low, is to iterate the eigenvector in Eq. (1) to find a more accurate value of the lowest eigenvalue.

Further, it is worth noting that if one is willing to perform repeated operations with high order matrices, as in the dynamic condensation method, in many cases it may be computationally more economical to perform an inverse iteration step[7,8,10,11] with the complete matrix equation rather than to compute an additional iterative step of dynamic condensation for refinement of eigenvectors and eigenvalues. In fact, for real symmetric matrices, efficient routines exist for determining eigenvalues without the corresponding eigenvectors in about one-fourth the time required for both the eigenvalues and eigenvectors.[12] One step of inverse iteration for each eigenvalue of interest can then be used to compute highly accurate eigenvectors.

The inverse iteration for the eigenvector y using the complete matrix equation is

$$[K-\Omega^2(n)M]y_{n+1}=My_n \tag{3}$$

where K and M are, respectively, the stiffness and matrices of the complete system, and $\Omega(n)$ is an approximate natural frequency squared obtained in the nth step of the iterative process. The ratio y_n/y_{n+1} when it is sufficiently converged leads to the natural frequency ω^2 closest to $\Omega^2(n)$. The iterated value of Ω^2 is given by

$$\Omega^2(n+1)-\Omega^2(n)\approx y_n(x_i)/y_{n+1}(x_i) \tag{4}$$

where $y_n(x_i)$ and $y_{n+1}(x_i)$ are corresponding elements of the eigenvectors y_n and y_{n+1}.

The economical way to solve Eq. (3) when y_n is given is to solve for y_{n+1} rather than calculating $(K-\omega^2 M)^{-1}$. The number of multiplications is $n^3/3$ for a matrix of order n.[8] On the other hand, dynamic condensation calls for inversion of an $n-m$ matrix requiring $(n-m)^3$ multiplications[8] plus a number of other operations with high-order matrices.

Inverse iteration is known to give a highly accurate eigenvector when ω^2 is close to the exact value for any given mode. The Rayleigh quotient then gives a correspondingly more accurate eigenvalue. Comparison of $n^3/3$ for inverse iteration with $(n-m)^3$ for matrix inversion, taking $m=4$, $n=48$ as in Paz's example, shows that inverse iteration requires 36,864 multiplications vs 85,184 for the matrix inversion in a step of dynamic condensation. For $m=n/2$, inverse iteration of the complete problem will require 8/3 times as many multiplication as the inversion in dynamic condensation. However, the multiplications and other operations with matrices of order $n/2$ in dynamic condensation will make the

effort required for the two methods about equal. In this regard, working with the eigenvalue equations in statically condensed form (e.g., Eqs. (7) of Ref. 6), so that reduced and eliminated degrees of freedom are coupled only dynamically, would reduce the number of matrix computations required in each step of Paz's dynamic condensation algorithm.

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Reply by Author to A. H. Flax

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THE writer wishes to thank Flax for his interest and comments on the paper.[19] Flax points out that the dynamic condensation method is a definite improvement over the static condensation method of Guyan[1] (Reference numbers refer to original citations in the paper, Ref. 19) and Irons[2] since the proposed method leads to a virtually exact solution of the generalized eigenproblem that arises in structural dynamics. He also recognizes that the dynamic condensation method improves both the eigenvalues and the eigenvectors and not just the eigenvectors, as is the case with the modified condensation method devised by Kidder,[7] Miller,[14,15] and others. Most important, the dynamic condensation method does not require the series expansion of the term $[\,[K_{ss}] - \omega^2 [M_{ss}]\,]^{-1}$ and the truncation of higher order terms, which poses the question of convergence of the series as discussed by Flax in Ref. 9.

Having recognized the merits of the proposed method, Flax states: "However, dynamic condensation greatly increases the amount of computation required, since the method involves repeated inversions of matrices of order equal to the number of degrees of freedom eliminated from the reduced problem." This statement constitutes an unfortunate misunderstanding of the paper and of the method. The dynamic condensation method requires no matrix inversion, but rather the application of the process known as the Gauss-Jordan elimination as it is routinely applied to the solution of a system of linear equations. This elementary process is carried out over the secondary coordinates to be eliminated. As indicated by Eq. (2) of the paper, this process immediately results in the transformation matrix $[\bar{T}]$ and the dynamic matrix $[\bar{D}]$ in Eq. (2). The simple multiplication and addition of matrices then gives the reduced stiffness, Eq. (4), and the reduced mass matrices, Eq. (5). Furthermore, there is no need to perform explicitly the product $[\bar{M}] = [T]^T [M] [T]$ which involves large matrices of dimensions equal to that of the original problem before reduction. By operating on the partition matrices of the matrices $[T]$ and $[M]$ a simple algorithm is obtained for the direct evaluation of the reduced mass matrix $[\bar{M}]$ which has the dimensions of the reduced problem.

In the illustrative example presented in the Note, the calculation of $[\bar{M}]$ requires the algebraic evaluation of the four elements of the symmetric matrix $[\bar{M}]$. Thus, the example of 48 degrees of freedom reduced to 4 degrees of freedom only requires the sequential applications of the Gauss-Jordan method in the linear symmetric system of 44 equations and the calculation of the elements of $[\bar{K}]$ and $[\bar{M}]$ of order 4 in addition to the eigensolution of four eigenproblems of order 4. These simple calculations result in almost exact eigensolutions for all four modes left in the reduced problem. In the illustrative example, the errors in the four lower eigenvalues obtained after reduction are 1.2%, 0.2%, 0.7%, and 3.5%. These values should be contrasted with the results produced by either static or modified condensation with errors respectively equal to 1.2%, 11.3%, 30.4%, and 36.8%. Furthermore, the application of a single iterative cycle of dynamic condensation results in the virtually exact eigensolution as indicated in Table 1 of the paper.[19]

The writer agrees with Flax in the need for critically comparing dynamic condensation with other methods of eigensolution such as inverse iteration or other variations such as the popular subspace iteration method. However, such comparisons can ony be made after an efficient algorithm is developed for the implementation of the dynamic condensation method. An overview of all the three methods of condensation—static, modified, and dynamic—is given in the previous paper.[17] This material is also presented with numerical examples and corresponding computer program in the second edition of the writer's textbook.[18]

As a final point, considering the fundamental importance that the solution of the eigenproblem has a structural dynamics, the writer agrees with other comments of Flax leading to further development and evaluation of the dynamic condensation method.

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